Proof of Implicit Function Theorem, continued:

Recall: We had obtained our W and U and had verified all properties. We now know that $\exists$ g, $g : W \to \mathbb{R}^n$.

We want g differentiable, with

$$g'(b) = -A_x^{-1} A_y$$

Step 2: (derivatives)

We had defined

$h : \mathbb{R}^2 \to \mathbb{R}^2$

by $h(x,y) = (f(x,y), y)$.

Furthermore, we know that h is invertible on U and h^{-1} is differentiable on $h(U)$.

We defined

$$W = \{y : (0,y) \in V\}.$$

This implies that

$$h(g(y), y)$$

$$= (f(x,y), y)$$

$$= (0, y) \quad \forall y \in W.$$

Then applying h^{-1},

$$(g(y), y) = h^{-1}(0, y).$$

Since h^{-1} is differentiable, we immediately obtain that g is differentiable.

Calculating $g'(b)$

Define $k: W \to \mathbb{R}^2$

$k(y) = (g(y), y)$.

We have

$f(k(y)) = f(g(y), y)$

If $y \in W$,

then by the chain rule,

$$D(f \circ k)(y)$$

$$= f'(k(y)) \cdot k'(y)$$

Then $D(f \circ k)(b)$

$$= f'(k(b)) \cdot k'(b)$$

$$= f'(g(b), b) \cdot k'(b)$$

$$= f'(a, b) \cdot k'(b)$$

But

$$f'(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

$$= (A_x, A_y)$$

What is $k'(y)$?

$$k'(y) = (g'(y), 1)$$

Taking the dot product,

$D(f \circ k)(b)$

$= (A_x, A_y) \cdot (g'(b), 1)$

$= A_x g'(b) + A_y$

But $D(f \circ k)(b)$ is zero since for all $y \in W$, we have $f \circ k(y)$

$= f(g(y), y)$

$= 0.$

Then solving for g', we get

$$g'(b) = -A_x^{-1} A_y . \quad \square$$

Example 1:

Let $f(x,y) =$

$$\sin(x^2y + y^3) + e^{xy} - 1$$

$$\frac{\partial f}{\partial y} = (x^2 + 3y^2)\cos(x^2y + y^3) + xe^{xy}$$

$$f(1,0) = 0$$

$$\frac{\partial f}{\partial y}(1,0)$$

$$= 2 \neq 0.$$

This says

$f(x,y) = 0$ can be solved

for y in terms of x

in a neighborhood about

$y = 0$. But try to

do this explicitly!

Integration in $\mathbb{R}^n$

The goal:

$$f : \mathbb{R}^n \to \mathbb{R}.$$

If we can define

$$\int_A f(x)\,dx \quad \text{for } A \subseteq \mathbb{R}^n,$$

is there a "fundamental theorem of calculus" that allows us to evaluate the integral?

Definition: (n-cube)

Let (a_i, b_i), $1 \leq i \leq n$, be intervals in $\mathbb{R}$.

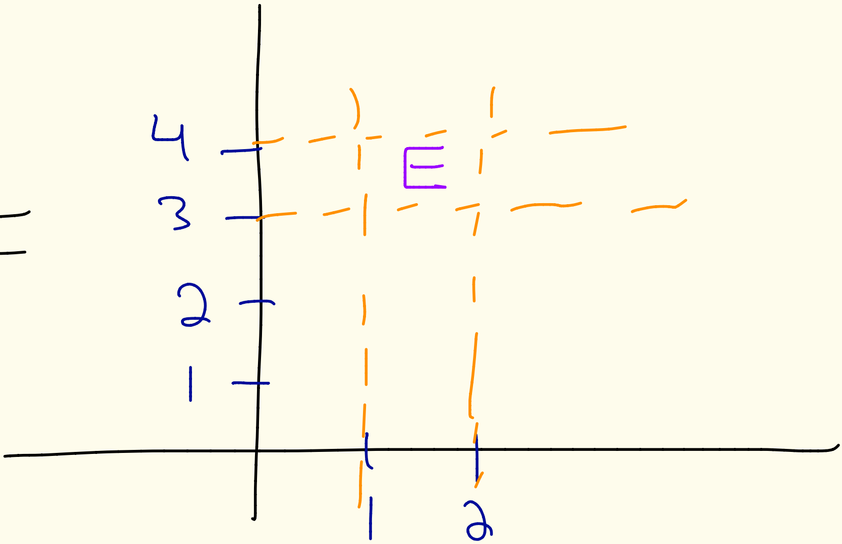
We define an open n-cube E to be

$$\prod_{i=1}^{n} (a_i, b_i)$$

$$= \{(x_1, x_2, \dots, x_n) \mid x_i \in (a_i, b_i) \; \forall i\}$$

$\underline{n=2}$ $\quad (1,2) \times (3,4)$

$=$



Remark: $\overline{E} = \prod_{i=1}^{n} [a_i, b_i]$.

Note also that with the metric $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, $(\mathbb{R}^n, d)$ satisfies the Heine-Borel theorem:

A subset $K \subseteq \mathbb{R}^n$ is compact if and only if K is closed and bounded

("bounded" means $\exists$ an n-cube $E = \prod_{i=1}^{n} (a_i, b_i)$ with $(a_i)_{i=1}^{n}$ and $(b_i)_{i=1}^{n}$ real numbers and $K \subseteq E$)

Definition: (integral in $\mathbb{R}^n$)

Let $E \subseteq \mathbb{R}^n$ be an n-cube.

Let $f: E \to \mathbb{R}$. Let

$E = \prod_{i=1}^{n} (a_i, b_i)$ and

$\{P_i\}_{i=1}^{n}$ partitions of

$\{(a_i, b_i)\}_{i=1}^{n}$, respectively.

If f is bounded on E,

define

$$U(f, \{P_i\}_{i=1}^{n})$$

$$\sum_{j_n=1}^{k_n} \cdots \sum_{j_2=1}^{k_2} \sum_{j_1=1}^{k_1} M_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} \prod_{i=1}^{n} \ell(x_{i_{j_i}}, x_{i_{j_i+1}})$$

where $M_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} = \sup_{x \in \prod_{i=1}^{n} (x_{i_{j_i}}, x_{i_{j_i+1}})} f(x)$

and

$$P_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,k_i}\}$$

where k_i = number of subdivisions.

Similarly, define

$$L(f, \{P_i\}_{i=1}^{n})$$

$$\sum_{j_n=1}^{k_n} \cdots \sum_{j_2=1}^{k_2} \sum_{j_1=1}^{k_1} m_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} \prod_{i=1}^{n} \ell(x_{i_{j_i}}, x_{i_{j_i+1}})$$

where $m_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} = \inf_{x \in \prod_{i=1}^{n} (x_{i_{j_i}}, x_{i_{j_i+1}})} f(x)$

Let $$L(f) = \sup_{\substack{\text{partitions} \\ \{P_i\}_{i=1}^n}} L(f, \{P_i\}_{i=1}^n)$$

and

$$U(f) = \inf_{\substack{\text{partitions} \\ \{P_i\}_{i=1}^n}} U(f, \{P_i\}_{i=1}^n)$$

We say f is

Riemann Integrable

on E if

$$\boxed{U(f) = L(f)}$$

In that case, we set

$\int_E f(x)\,dx$ to be

the common value of $U(f)$ and $L(f)$.